Minimal Lipschitz Extensions to differentiable functions

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Abstract

We generalize the Lipschitz constant to Whitney's functions and prove that any Whitney's function defined on a non-empty subset of $\mathbb{R}^n$ extends to a Whitney's function of domain $\mathbb{R}^n$ with the same constant. The proof uses an argument which refines the one used by Kirszbraun in the continuous case and, for this reason, holds only for $\mathbb{R}^n$ equipped with the euclidean norm. This constant is exactly the Lipschitz constant of the gradient of the extension and, therefore, this extension is minimal.

We continue the paper with a first approach of the absolutely minimal Lipschitz extension problem in the differentiable case.

1 Introduction

We prove in this paper that any 1-Whitney's function defined on a non-empty subset of $\mathbb{R}^n$ extends to a differentiable function whose Lipschitz constant of the differential is minimal. It is, as far as we know, the first result concerning the minimal Lipschitz extension problem in the multivariate differentiable case. The construction of the extension involved in the proof is also the first one which differs from the original Whitney's construction [13] and of its variant by Hestenes [9].

Results of Whitney [13] and McShane [12] have opened the door to the formulation, the connexion with PDE $\Delta_\infty u = 0$ and the resolution of the absolutely minimal Lipschitz extension problem in the continuous case by Aronsson [1] and Jensen [10]. The results of this paper open the similar door in the differentiable case.

The paper is organized as follows. Section 2 is concerned by definitions and known results concerning the Lipschitz extension problem in the differentiable case. Section 3 is concerned by the minimal Lipschitz extension problem and contains the main result. Section 4 deals with the absolutely minimal problem.

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2 Lipschitz extensions

A $m-$Whitney's function is a function $T : a \in dom(T) \subset \mathbb{R}^n \to T_a \in \mathcal{P}_m(\mathbb{R}^n, \mathbb{R}^d)$ where $\mathcal{P}_m(\mathbb{R}^n, \mathbb{R}^d)$ denotes the space of polynomials functions $P : x \in \mathbb{R}^n \to P(x) \in \mathbb{R}^d$ of total degree at most m.

A function (usual or $m-$Whitney's) is said to be total if its domain is $\mathbb{R}^n$.

A total $\mathcal{C}^m$-function u is said to be an extension of a m-Whitney's function T if for any $a \in dom(T)$, the m-Taylorian expansion of u at a coincides with T_a. H. Whitney [13] gave necessary and sufficient conditions on a m-Whitney's function of closed domain to extend to a total $\mathcal{C}^m$-function.

We say that a total $\mathcal{C}^m$-function u is m-Lipschitzian if its m^{th} derivative $D^m(u)$ is Lipschitz that is $L^0(D^m(u)) < +\infty$ where $L^0(D^m(u))$ is the usual Lipschitz constant of $D^m(u)$:

$$L^0(D^m(u)) := \sup_{a \neq b \in \mathbb{R}^n} \frac{||| D^m(u)(a) - D^m(u)(b) |||}{\| b - a \|} \tag{2.1}$$

where $\| \ \|$ denotes any norm on $\mathbb{R}^n$ and $||| \ |||$ any norm on the space of the m-linear symmetric applications from $\mathbb{R}^n$ to $\mathbb{R}^d$. G. Glaeser [7], by a sharp examination of the Whitney's construction, gave a necessary and sufficient condition for a Whitney's function T to extend to a total m-Lipschitzian function. This condition is $K_G^m(T) < +\infty$ where

$$K_G^m(T) := \sup_{a \neq b \in dom(T)} \sup_{x \in \mathbb{R}^n} \frac{\| T_a(x) - T_b(x) \|}{\| b - a \| (\| x - a \|^m + \| x - b \|^m)} \ . \tag{2.2}$$

Precisely G. Glaeser showed that any T satisfying $K_G^m(T) < +\infty$ can be extended to a total m-Lipschitzian function u such that $L^0(D^m(u)) \leq C_{m,n} K_G^m(T)$ where $C_{m,n}$ is an absolute constant which depends only on m and n.

Chr. Coatmelec proved that this condition can also be written $K_W^m(T) < +\infty$ or $K_C^m(T) < +\infty$ where

$$K_W^m(T) := \sup_{a \neq b \in dom(T)} \sup_{k=0,\ldots,m} \frac{\| D^k T_a(b) - D^k T_b(b) \|}{\| b - a \|^{m+1-k}} (m-k)! \ , \tag{2.3}$$

is issued from Whitney's original paper and

$$K_C^m(T) := \sup_{a \neq b \in dom(T)} \sup_{x \in Diam(a,b)} \frac{\| T_a(x) - T_b(x) \|}{\| b - a \|^{m+1}} \ , \tag{2.4}$$

where $Diam(a, b)$ denotes the ball of diameter $[a, b]$, is issued from Coatmelec's one [5].

Remark 2.1. When $dom(T)$ has a unique element we set $K_W^m(T) = K_G^m(T) = K_C^m(T) := 0$.

From now on we say that a m-Whitney's function T is Lipschitzian if it can be extended to a total m-Lipschitzian function that is if $K_G^m(T) < +\infty$ (or equivalently $K_W^m(T) < +\infty$ or $K_C^m(T) < +\infty$).

Note here that any Lipschitzian m-Whitney's function T has a unique extention $\bar{T}$ to the closure $\overline{dom}(T)$ of $dom(T)$; this extension satisfies $K_W^m(\bar{T}) = K_W^m(T)$.

It follows from Glaeser's result that total Lipschitzian m-Whitney's functions and total m-Lipschitzian functions can be canonically identified as follows. To each total m-Whitney's function u is canonically associated the total Lipschitzian m-Whitney's function $x \in \mathbb{R}^n \to U_x \in \mathcal{P}_m(\mathbb{R}^n, \mathbb{R}^d)$ where U_x is the m-Taylorian expansion of u at x. Conversely to each total Lipschitzian m-Whitney's function U is canonically associated the m-Lipschitzian function u defined by $u(x) := U_x(x)$. It follows from above that these two canonical mappings are reciprocal.

3 Minimal Lipschitz extensions

Definition 3.1. Let T be any Lipschitzian m-Whitney's function. We say that a total m-Lipschitzian function u is a minimal extension of T if u extends T and if for any total m-Lipschitzian function v which extends T we have

$$L^0(D^m(v)) \geq L^0(D^m(u)) \ .$$

Note that, in the case $m = 0$, the above definition of Lipschitz minimality contains (but does not reduces to) the classical one. This definition allows to enlarge the problem of the minimal Lipschitz extension in cases where the classical minimal problem has no solution (see the counter-example of [6], p. 202). Remark here that Glaeser's result applied for $m = 0$ insures that any partial Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}^d$ can be extended to a total Lipschitz function.

In the case $m = 0$, $d = 1$ (any n) it is known for a long time [[13], footnote p. 63], [12], that the problem of existence of minimal Lipschitz extensions has a positive solution. It is also the case for $d > 1$, $\mathbb{R}^n$, $\mathbb{R}^d$ euclidean : this result is due to Kirszbraun [11]. The proof of the main theorem of this paper uses an argument which refines the one used by Kirszbraun.

In the case $n = 1$, $d = 1$ (any m), it will be seen in section 4 that a stronger problem has been complety positively solved by G. Glaeser [8].

From now on in this section 3 we fix $m = 1$, $d = 1$ (any n). The space $\mathbb{R}^n$ is equipped with the standard scalar inner product $< .,. >$ and the associated euclidean norm $\| \, . \, \| = < .,. >^{1/2}$. Via the scalar product we identify vectors of $\mathbb{R}^n$

and linear forms on $\mathbb{R}^n$, in particular gradient and differential. For typographical convenience $\| b-a \|$ is denoted ab.

For any 1-Whitney's function T we write $T_a(x) = u_a + < D_a u, x-a >$, $a \in dom(T)$, $x \in \mathbb{R}^n$.

For any Lipschitzian 1-Whitney's function T we define

$$L^1(T) := 0 \text{ if } dom(T) \text{ has a single element}$$

and

$$L^1(T) := \sup_{a \neq b \in dom(T)} \left(\sqrt{A_{a,b}^2 + B_{a,b}^2} + \mid A_{a,b} \mid\right) \text{ if not }, \tag{3.1}$$

where

$$A_{a,b} := \frac{2(u_a - u_b) + < D_a u + D_b u, b-a >}{ab^2} ,$$

$$B_{a,b} := \frac{\| D_a u - D_b u \|}{ab} .$$

Proposition 3.2. *We have*

$$L^1(T) = 2 \sup_{a \neq b \in dom(T)} \sup_{x \in \mathbb{R}^n} \frac{T_a(y) - T_b(y)}{ay^2 + by^2} = 2 \sup_{a \neq b \in dom(T)} \sup_{x \in Diam(a,b)} \frac{T_a(y) - T_b(y)}{ay^2 + by^2} ,$$

where $Diam(a,b)$ denotes the ball of center $(a+b)/2$ and of radius $ab/2$.

Proof. For any $a \neq b \in dom(T)$ and $y \in \mathbb{R}^n$ we set

$$g_{a,b}(y) := \frac{T_a(y) - T_b(y)}{ay^2 + by^2} .$$

It is sufficient to prove that for any $a \neq b \in dom(T)$

$$\sup_{x \in \mathbb{R}^n} \mid g_{a,b}(x) \mid = \frac{1}{2}\left(\sqrt{A_{a,b}^2 + B_{a,b}^2} + \mid A_{a,b} \mid\right) ,$$

and that the supremum is attained inside $Diam(a,b)$.
For any $x \in \mathbb{R}^n$ we set

$$t := \frac{2}{ab}\left(x - \frac{a+b}{2}\right) .$$

We have

$$T_a(x) - T_b(x) = \frac{ab^2}{2} A_{a,b} + \frac{ab}{2} < D_a u - D_b u, t > ,$$

and

$$ax^2 + bx^2 = \frac{ab^2}{2}(1 + t^2) .$$

Therefore

$$\sup_{x\in\mathbb{R}^n} \mid g(x) \mid = \sup_{t\in\mathbb{R}^n} \frac{\mid A_{a,b} + < D_a u - D_b u, t > /ab \mid}{1 + \parallel t \parallel^2}$$

First case : $D_a u - D_b u = 0$. The maximum in t is attained for $t = 0$ and

$$\sup_{x\in\mathbb{R}^n} \mid g_{a,b}(x) \mid = \mid A_{a,b} \mid \ .$$

Second case : $D_a u - D_b u \neq 0$. We set

$$v_1 := \frac{D_a u - D_b u}{\parallel D_a u - D_b u \parallel}$$

and choose $v_2, ..., v_n \in \mathbb{R}^n$ with $\parallel v_k \parallel^2 = 1$ and $< v_k, v_l >= 0$ for $k \neq l$, and $\alpha, \beta_2, ..., \beta_n \in \mathbb{R}$ such that

$$t = \alpha v_1 + \sum_{k=2}^{n} \beta_k v_k \ .$$

We have $1 + \parallel t \parallel^2 = 1 + \alpha^2 + \sum_{k=2}^{n} \beta_k^2$, and $\frac{1}{ab} < D_a u - D_b u, t >= \alpha B_{a,b}$. Therefore

$$\sup_{x\in\mathbb{R}^n} \mid g_{a,b}(x) \mid = \sup_{\alpha\in\mathbb{R}} \frac{\mid A_{a,b} + \alpha B_{a,b} \mid}{1 + \alpha^2} \ .$$

It is now elementary to prove that

$$\sup_{\alpha\in\mathbb{R}} \frac{\mid A_{a,b} + \alpha B_{a,b} \mid}{1 + \alpha^2} = \frac{1}{2}(\sqrt{A_{a,b}^2 + B_{a,b}^2} + \mid A_{a,b} \mid)$$

and that the supremum is attained inside the interval $[-1, 1]$. □

Remark 3.3. Applying Glaeser's result [8] in the case $m = 1$, we can, when $dom(T) = \{a, b\}$, exactly compute the unique minimal extension u on $[a, b]$. We find $L^0(u^1) = (A_{a,b}^2 + B_{a,b}^2)^{1/2} + \mid A_{a,b} \mid$ and we can moreover verify that

$$L^0(u^1) = 2 \sup_{x\in\mathbb{R}} \frac{\mid T_a(x) - T_b(x) \mid}{ax^2 + bx^2} = 2 \sup_{x\in[a,b]} \frac{\mid T_a(x) - T_b(x) \mid}{ax^2 + bx^2} \ .$$

Remark 3.4. It can be checked that $L^1(T)$, $K_W^1(T)$, $K_G^1(T)$, $K_C^1(T)$ are equivalent and that T extends to a unique $\bar{T}$ defined on $\overline{dom}(T)$ satisfying $L^1(\bar{T}) = L^1(T)$. So, in the following, we can assume without loss generality that T has a closed domain.

Proposition 3.5. *Let u be a total 1-Lipschitzian function and U its associated Lipschitzian 1-Whitney's function. Then*

$$L^1(U) = L^0(Du) \ .$$

Proof. For any x,$y \in \mathbb{R}^n$ we set $U_x(y) := u_x + < D_x u, y - x >$. We can write

$$u_y - u_x = \int_0^1 < D_{x+t(y-x)} u, y - x > dt \ .$$

For any x,y, $z \in \mathbb{R}^n$ we have

$$\begin{aligned} \mid U_x(y) - u_y \mid &= \mid u_x - u_y + < D_x u, y - x > \mid \\ &= \mid \int_0^1 < D_{x+t(y-x)} u - D_x u, y - x > dt \mid \\ &\leq \int_0^1 \mid < D_{x+t(y-x)} u - D_x u, y - x > \mid dt \\ &\leq \int_0^1 \|(D_{x+t(y-x)} u - D_x u)\| \|y - x\| dt \\ &\leq L^0(Du) \|y - x\|^2 \int_0^1 t dt \\ &\leq \frac{1}{2} L^0(Du) \|y - x\|^2, \end{aligned} \tag{3.2}$$

and, therefore,

$$\begin{aligned} \mid U_x(z) - U_y(z) \mid &\leq \mid U_x(z) - u_z \mid + \mid u_z - U_y(z) \mid \\ &\leq \frac{1}{2} L^0(Du)(\|x - z\|^2 + \|y - z\|^2), \end{aligned} \tag{3.3}$$

that is $L^1(U) \leq L^0(Du)$.
Conversely, by proposition 3.2, we have

$$2 \sup_{z \in \mathbb{R}^n} \frac{\mid T_x(z) - T_y(z) \mid}{xz^2 + yz^2} = \sqrt{A_{x,y}^2 + B_{x,y}^2} + \mid A_{x,y} \mid \geq B_{x,y} \text{ for any } x \neq y \in \mathbb{R}^n \ ,$$

that is $L^1(U) \geq L^0(Du)$.

□

We are now ready to state and prove the main result of this paper.

Theorem 3.6. *Let T be a Lipschitzian 1-Whitney's function. Then T extends to a total Lipschitzian 1-Whitney's function U such that $L^1(U) = L^1(T)$. Moreover the total 1-Lipschitzian function $x \rightarrow U_x(x)$ associated with U is a minimal 1-Lipschitzian extension of T.*

Proof. Let us set $K := \frac{1}{2}L^1(T)$, $S := dom(T)$. By transfinite induction (that is by Zorn's lemma) it is sufficient to prove that for any $x \in \mathbb{R}^n - S$ there exist $u_x \in \mathbb{R}$, $D_x u \in \mathbb{R}^n$ such that

$$-K \leq \frac{T_x(y) - T_a(y)}{xy^2 + ay^2} \leq K \text{ , for any } a \in S \text{ , } y \in \mathbb{R}^n \text{ ,} \tag{3.4}$$

where $T_x(y) := u_x + < D_x u, y - x >$.

If $K = 0$ then the 1-Whitney's function T is constant on $dom(T)$ and we can (and must) extend T on $\mathbb{R}^n$ by this constant function. So, from now on, we assume that $K \neq 0$.

First step : Elimination of y.
The second inequality of 3.4 can be written

$$0 \leq g_a(y) \text{ for any } a \in S \text{ , } y \in \mathbb{R}^n \text{ .} \tag{3.5}$$

where

$$g_a(y) := T_a(y) - T_x(y) + K(xy^2 + ay^2) \text{ .}$$

Since function $y \to g_a(y)$ is quadratic and $K > 0$, the minimum in y is attained for $y_0 \in \mathbb{R}^n$ which realizes $D_{y_0} g_a = 0$ that is

$$y_0 = \frac{1}{2}(x + y) + \frac{1}{4K}(D_x u - D_a u) \text{ .}$$

So, after elementary calculations, condition 3.5 becomes equivalent to : for any $a \in S$

$$u_x \leq u_a + \frac{1}{2} < D_a u + D_x u, x - a > + \frac{K}{2} ax^2 - \frac{1}{8K} \parallel D_a u - D_x u \parallel^2 \text{ .} \tag{3.6}$$

Similarly, using the first inequality of 3.4, we obtain the following condition : for any $a \in S$

$$u_b + \frac{1}{2} < D_b u + D_x u, x - b > - \frac{K}{2} bx^2 + \frac{1}{8K} \parallel D_b u - D_x u \parallel^2 \leq u_x \text{ .} \tag{3.7}$$

So condition 3.4 is equivalent to both condition 3.6 and 3.7 where y has disappeared.

Second step : Elimination of u_x.
By examination of conditions 3.6 and 3.7 we see that u_x exists only if

$$\begin{gathered} u_b + \tfrac{1}{2} < D_b u + D_x u, x - b > - \tfrac{K}{2} bx^2 + \tfrac{1}{8K} \parallel D_b u - D_x u \parallel^2 \\ \leq \\ u_a + \tfrac{1}{2} < D_a u + D_x u, x - a > + \tfrac{K}{2} ax^2 - \tfrac{1}{8K} \parallel D_a u - D_x u \parallel^2 \\ \text{for any } a, b \in S \text{ .} \end{gathered} \tag{3.8}$$

But, conversely, if condition 3.4 holds we infer

$$min \leq max \tag{3.9}$$

where

$$min := \sup_{b \in S}(u_b + \frac{1}{2} < D_b u + D_x u, x - b > - \frac{K}{2} bx^2 + \frac{1}{8K} \parallel D_b u - D_x u \parallel^2) \ ,$$

and

$$max := \inf_{a \in S}(u_a + \frac{1}{2} < D_a u + D_x u, x - a > + \frac{K}{2} ax^2 - \frac{1}{8K} \parallel D_a u - D_x u \parallel^2) \ .$$

It follows that the existence of u_x is a consequence of the existence of $D_x u$ satisfying (3.8) : we can take for u_x any number between min and max.

So, at this stage of the proof, we have eliminated $y \in \mathbb{R}^n$ and $u_x \in \mathbb{R}$ and we have only to prove the existence of a $D_x u \in \mathbb{R}^n$ which satisfies condition (3.8).

Third step : Elimination of $D_x u$ (geometrical formulation).
Writing condition (3.8) under the form $Q \leq 0$, we see that Q is a quadratic polynomial of $D_x u$. Writing this quadratic polynomial under canonical form, condition (3.8) becomes, after tedious but elementary calculations,

$$\|D_x u - V_{a,b}\|^2 \leq \alpha_{a,b} + \beta_{a,b} \ , \text{ for any } a, b \in S \ , \tag{3.10}$$

where

$$V_{a,b} := \frac{1}{2}(D_a u + D_b u) + K(b - a),$$

$$\alpha_{a,b} := \frac{1}{2}(4K(2(u_a - u_b) + < D_a u - D_b u, b - a >) - \parallel D_a u - D_b u \parallel^2 + 4K^2 ab^2) \ ,$$

$$\beta_{a,b} := \| \frac{1}{2}(D_a u - D_b u) + K(2x - a - b) \|^2 \ .$$

Using the definition of $L^1(T)$ we have $(A_{a,b}^2 + B_{a,b}^2)^{1/2} + \mid A_{a,b} \mid \leq 2K$, and therefore

$$0 \leq -4K \mid A_{a,b} \mid - B_{a,b}^2 + 4K^2 \ . \tag{3.11}$$

Since $\alpha_{a,b}$ can be writen $\alpha_{a,b} = \frac{1}{2}(4KA_{a,b} - B_{a,b}^2 + 4K^2)ab^2$, it follows that $\alpha_{a,b} \geq 0$.

Then, setting $r_{a,b} := \sqrt{\alpha_{a,b} + \beta_{a,b}}$, condition (3.8) becomes

$$\|D_x u - V_{a,b}\|^2 \leq r_{a,b}^2 \ , \text{ for all } a, b \in S \ . \tag{3.12}$$

In other words, denoting by $B_{a,b}$ the ball of center $V_{a,b}$ and of radius $r_{a,b}$, the original problem reduces to :

$$\cap_{a,b \in S} B_{a,b} \neq \emptyset \ , \tag{3.13}$$

or, using compactness,

$$\cap_{a,b\in F} B_{a,b} \neq \emptyset, \text{ for any finite non empty subset } F \text{ of } S \ . \tag{3.14}$$

From now on, by 3.14, we shall assume without loss of generality that S is finite.

Fourth step : elimination of $D_x u$(algebraic formulation).
For any $a, b, c, d \in S$ we set

$$\Phi(a,b,c,d) := r_{a,b}^2 + r_{c,d}^2 - \| V_{a,b} \|^2 - \| V_{c,d} \|^2 \ .$$

Expanding again $V_{a,b}$ and $V_{c,d}$, we can write

$$\Phi(a,b,c,d) = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \ ,$$

where

$$\Phi_1 = \alpha_{a,d} + \alpha_{c,b} \ ,$$

$$\Phi_2 = 2K^2(ax^2 + bx^2 + cx^2 + dx^2) \ ,$$

$$\begin{aligned}\Phi_3 \quad = \quad & 2(< D_a u, K(x-d) > + < D_b u, K(c-x) > \\ & + < D_c u, K(x-b) > + < D_d u, K(a-x) >),\end{aligned}$$

and

$$\Phi_4 = - < D_a u, D_d u > - < D_c u, D_b u > \ .$$

Since $\Phi_1 = \alpha_{a,d} + \alpha_{c,b} = \frac{1}{2}((4KA_{a,d} - B_{a,d}^2)ad^2 + (4KA_{c,b} - B_{c,b}^2)cb^2)$ we infer, using 3.11, that

$$\Phi_1 \geq -2K^2(ad^2 + cb^2) \ .$$

It follows that

$$\begin{aligned}\Phi_1 + \Phi_2 \quad & \geq \quad 2K^2(ax^2 + bx^2 + cx^2 + dx^2 - ad^2 - cb^2) \\ & \geq \quad -4K^2(< x-a, d-x > + < x-c, b-x >),\end{aligned}$$

and, therefore, that

$$\begin{aligned}\Phi(a,b,c,d) \quad \geq \quad & -4(< \frac{1}{2}D_a u + K(x-a), \frac{1}{2}D_d u + K(d-x) > \\ & + < \frac{1}{2}D_c u + K(x-c), \frac{1}{2}D_b u + K(b-x) >) \ .\end{aligned} \tag{3.15}$$

Let us now show that if $r_{a,b} = r_{c,d} = 0$ then $V_{a,b} = V_{c,d}$.
Setting $P := D_a u + 2K(x-a)$ and $Q := D_c u + 2K(x-c)$.
Condition $r_{a,b} = r_{c,d} = 0$ implies
$P = D_b u - 2K(x-b)$, $Q = D_d u - 2K(x-d)$
and, therefore,
$V_{a,b} = \frac{1}{2}D_a u + K(x-a) + \frac{1}{2}D_b u - K(x-b) = P$

$V_{c,d} = \frac{1}{2}D_c u + K(x-c) + \frac{1}{2}D_d u - K(x-d) = Q.$
We can write
$r_{a,b}^2 + r_{c,d}^2 - \parallel V_{a,b} - V_{c,d} \parallel^2 = \Phi(a,b,c,d) + 2 < P, Q >.$

By (3.15) we have

$$\begin{aligned}\Phi(a,b,c,d) \quad &\geq \quad -4(< \frac{1}{2}D_a u + K(x-a), \frac{1}{2}D_d u + K(d-x) > \\ &\qquad + < \frac{1}{2}D_c u + K(x-c), \frac{1}{2}D_b u + K(b-x) >),\end{aligned}$$

that is $\Phi(a,b,c,d) \geq -2 < P, Q >$.
Therefore $r_{a,b}^2 + r_{c,d}^2 - \parallel V_{a,b} - V_{c,d} \parallel^2 \geq 0$.
Since $r_{a,b} = r_{c,d} = 0$ we obtain $V_{a,b} = V_{c,d}$.
In other words, we conclude that balls $B_{a,b}$ of radius 0 have the same center.

For any $\lambda \geq 0$ let us denote by $B_{a,b}(\lambda)$ the ball of center $V_{a,b}$ and of radius $\lambda r_{a,b}$. Since balls $B_{a,b}$ of radius 0 have the same center, it follows that all balls $B_{a,b}(\lambda)$ intersect for λ sufficiently large. For the smallest λ for which $\cap_{a,b\in S} B_{a,b}(\lambda) \neq \emptyset$, this intersection countains a single element r and this element belongs to the convex hull of the set E of those $V_{a,b}$ such that $\|r - V_{a,b}\| = \lambda r_{ab}$ (see [[6], p.199] for a proof). It follows that we can write

$$r := \sum_{(a,b)\in E_0} \xi_{a,b} V_{a,b} \ , \quad \sum_{(a,b)\in E_0} \xi_{a,b} = 1 \ . \tag{3.16}$$

where $E_0 := \{(a,b) \in S^2 : V_{a,b} \in E\}$, $\xi_{a,b} \geq 0$, $\sum_{(a,b)\in E_0} \xi_{a,b} = 1$.

Now we have to prove that $\lambda \leq 1$.

Using (3.16) we obtain

$$\sum_{(a,b),(c,d)\in E_0} \xi_{a,b}\xi_{c,d} < r - V_{a,b}, r - V_{c,d} >= 0 \ . \tag{3.17}$$

For any $a, b, c, d \in S$ we have

$$\|V_{c,d} - V_{a,b}\|^2 = \|r - V_{a,b}\|^2 + \|r - V_{c,d}\|^2 - 2 < r - V_{a,b}, r - V_{c,d} > \ .$$

For any $(a,b), (c,d) \in E_0$ we have therefore

$$\|V_{c,d} - V_{a,b}\|^2 = \lambda^2 r_{a,b}^2 + \lambda^2 r_{c,d}^2 - 2 < r - V_{a,b}, r - V_{c,d} > \ . \tag{3.18}$$

Multiplying (3.18) by $\xi_{a,b}\xi_{c,d}$ adding and using (3.17) we obtain

$$0 = \sum_{(a,b),(c,d)\in E_0} \xi_{a,b}\xi_{c,d}(-\|V_{c,d} - V_{a,b}\|^2 + \lambda^2 r_{a,b}^2 + \lambda^2 r_{c,d}^2) \ . \tag{3.19}$$

Setting

$$\Delta := \sum_{(a,b),(c,d)\in E_0} \xi_{a,b}\xi_{c,d}(-\|V_{c,d} - V_{a,b}\|^2 + r_{a,b}^2 + r_{c,d}^2) \ , \tag{3.20}$$

and substracting (3.19) from (3.20) we obtain

$$\Delta = (1-\lambda^2) \sum_{(a,b),(c,d)\in E_0} \xi_{a,b}\xi_{c,d}(r_{a,b}^2 + r_{c,d}^2) \ .$$

If $r_{a,b} = 0$ for any $(a,b) \in E_0$ then, using the definition of E_0 and the fact -already proved- that all balls $B_{a,b}$, $(a,b) \in E_0$ have in this case the same center, we see that $D_x u$ can -and must- be choosen to be this center.

Else condition $\lambda \leq 1$ is equivalent to condition $\Delta \geq 0$.

Final step : Verification of condition $\Delta \geq 0$.
Since

$$\| \ r \ \|^2 = \sum_{(a,b),(c,d)\in E_0} \xi_{a,b}\xi_{c,d} < V_{a,b}, V_{c,d} > \ , \tag{3.21}$$

we can write

$$\Delta = 2 \ \| \ r \ \|^2 + \sum_{(a,b),(c,d)\in E_0} \xi_{a,b}\xi_{c,d}\Phi(a,b,c,d) \ .$$

Let us set

$$X := \sum_{(a,b)\in E_0} \xi_{a,b}(\frac{1}{2}D_a u + K(x-a)) \ , \ Y := \sum_{(a,b)\in E_0} \xi_{a,b}(\frac{1}{2}D_b u + K(b-x)) \ .$$

Now, using (3.21) and the definition of $V_{a,b}$ and $V_{c,d}$, a computation shows that

$$2 \ \| \ r \ \|^2 = 2\|X+Y\|^2 \ . \tag{3.22}$$

Using inequality (3.15) we infer

$$\begin{aligned} \Phi(a,b,c,d) \quad &\geq \quad -4(< \frac{1}{2}D_a u + K(x-a), \frac{1}{2}D_d u + K(d-x) > \\ &\qquad + < \frac{1}{2}D_c u + K(x-c), \frac{1}{2}D_b u + K(b-x) >) \ . \end{aligned} \tag{3.23}$$

It follows from that

$$\sum_{(a,b),(c,d)\in E_0} \xi_{a,b}\xi_{c,d}\Phi(a,b,c,d) \geq -8 < X, Y > \ . \tag{3.24}$$

In definitive, using (3.22) and (3.24) we obtain that

$$\Delta \geq 2 \ \| \ X+Y \ \|^2 - 8 < X, Y > = 2 \ \| \ X-Y \ \|^2 \geq 0 \ ,$$

which is the desired result.

To finish the proof of the theorem let v be any 1-Lipschitzian total function extending T. Using proposition 3.5 we have $L^0(Dv) = L^1(V)$ where V denotes the 1-Whitney's function associated to v. Since V extends T we have $L^1(V) \geq L^1(T)$ because *sup* are taken on $\mathbb{R}^n$ which is larger than $dom(T)$. Using the first part of the theorem and proposition 3.5 again we have $L^0(Du) = L^1(U) = L^1(T)$. It follows that $L^0(Dv) \geq L^0(Du)$. $\square$

3.1 Questions of uniqueness.

We finish this section 3 by some words concerning the uniqueness. Elementary examples in one variable ($n = 1$) show that the minimal extensions are not in general unique. Proposition 3.9 shows however that all extensions coincide for some points. Note here that G. Aronsson has characterized the set of uniqueness in the continuous case [[1], theorem 2].

We considere again a Lipschitzian 1-Whitney's function T with $S := dom(T)$ and $K := \frac{1}{2}L^1(T)$ and we define the total upper and lower extensions W^+ and W^- by

$$W^+(y) := \inf_{b\in S}(T_b(y) + kby^2) \text{ , for any } y \in \mathbb{R}^n \text{ ,}$$

$$W^-(y) := \sup_{a\in S}(T_a(y) - kay^2) \text{ , for any } y \in \mathbb{R}^n \text{ .}$$

Remark 3.7. Upper and lower total extensions W^+ and W^- are continuous but, in general, not differentiable.

Lemma 3.8. *Let u be a total 1-Lipschitzian minimal extension of T. Then*

$$W^-(y) \leq u(y) \leq W^+(y) \textit{ for any } y \in \mathbb{R}^n \text{ .}$$

Proof. Let U be the 1-Whitney's function associated to u. Since $L^1(U) = L^1(T) = 2K$ we have, by proposition 3.2 applied to U :

$$U_a(z) - K(az^2 + yz^2) \leq U_y(z) \text{ , for any } y, z \in \mathbb{R}^n \text{ , } a \in S \text{ ,} \tag{3.25}$$

and

$$U_y(z) \leq U_b(z) + K(bz^2 + yz^2) \text{ , for any } y, z \in \mathbb{R}^n \text{ , } b \in S \text{ .} \tag{3.26}$$

Since 3.25 and 3.26 hold in particular for $a, b \in S$ we obtain by taking $z = y$ in 3.25, 3.26 and using the fact that U extends T

$$T_a(y) - Kay^2 \leq u(y) \leq T_b(y) + Kby^2 \text{ , for any } a, b \in S, y \in \mathbb{R}^n \text{ ,}$$

which implies $W^-(y) \leq u(y) \leq W^+(y)$, for any $y \in \mathbb{R}^n$. $\square$

It follows from Lemma 3.8 that we can hope uniqueness on the set $E_0 := \{x \in \mathbb{R}^n - S : W^-(x) = W^+(x)\}$. The case $K = 0$ being trivial we assume $K > 0$ in the following.

Proposition 3.9. *Let us assume that S is compact. Then all minimal extensions of T coincide on E_0.*

Proof. Let $x \in E_0$. Since functions $a \in S \to T_a(x) - Kxa^2$, $b \in S \to T_b(x) + Kxb^2$ are continuous, there exists, by compactness of S, $a, b \in S$ such that
$T_a(x) - Kxa^2 = T_b(x) + Kxb^2$.

Now let u be any minimal 1-Lipschitzian extension of T and U its associated 1-Whitney's function : $U_x(y) := u(x) + < Du(x), y - x >$, $x, y \in \mathbb{R}^n$.

Using Lemma 3.8 we have $u(x) = T_a(x) - Kxa^2 = T_b(x) + Kxb^2$.Writing
$T_a(x) = t_a + < D_a t, x - a >$ and $T_b(x) = t_b + < D_b t, x - b >$,
using inequality 3.26 and equality $u(x) = W^+(x)$, we obtain for any $y \in \mathbb{R}^n$

$$U_x(y) - u(x) \leq T_b(y) + K(by^2 + xy^2) - W^+(x) = T_b(y) + K(by^2 + xy^2) - T_b(x) - Kbx^2.$$

Since $T_b(y) = T_b(x) + < D_b t, y - x >$, we obtain after simplifications

$$< Du(x), y - x > \leq < D_b t, y - x > + K(2 < b - x, x - y > + xy^2) \ .$$

Setting $\tilde{D}_x t := D_b t + 2K(x - b)$, we obtain therefore

$$< Du(x) - \tilde{D}_x t, y - x > \leq 2Kxy^2 \text{ for any } y \in \mathbb{R}^n \ .$$

Now, towards a contradiction, let us suppose that $V := D_x u - \tilde{D}_x u \neq 0$. Setting $y - x = \epsilon V / \parallel V \parallel$, with $\epsilon > 0$ we obtain $\parallel V \parallel \leq K\epsilon$ from which we infer $V = 0$ by letting ϵ tend to 0.
We have therefore proved that $u(x)$ and $Du(x)$ take values which depend only on $T_a(x)$ and $T_b(x)$. In other words all minimal extension agree on E_0. Note here that it can easily be shown that $\tilde{D}_x t := D_a t - 2K(x - a) = V_{b,a}$, where $V_{b,a} := \frac{1}{2}(D_a t + D_b t) + K(a - b)$ has already appeared in the proof of theorem 3.6.

□

4 Absolutely minimal Lipschitz extensions

In the case $m = 0$, $d = 1$, $\mathbb{R}^n$ euclidean (any n), the problem of the existence and of the uniqueness of absolutely minimal Lipschitz extensions, settled by G. Aronsson, has been completely positively solved by Aronsson [1] and Jensen [10] (see [4] for a recent tour on the subject).

Let us show in this section 4 that the problem has also been completely positively solved in the case $m = d = 1$, (any m) as a direct consequence of theorem of G. Glaeser [8].

In the case $m = d = 1$, G. Glaeser proved that, given a m-Whitney's function of $dom(T) = \{a, b\}$ there exists a unique function u having the following properties :
1) u is m-times differentiable on $[a, b]$,
2) u extends T,
3) $u^{(m)}$ is absolutely continuous on $[a, b]$,
4)
$$ess \sup_{x \in [a,b]} \mid v^{(m+1)}(x) \mid \geq ess \sup_{x \in [a,b]} \mid u^{(m+1)}(x) \mid$$
for any v satisfying 1),2),3).
This function is a perfect m-spline that is a $\mathcal{C}^m$-function on $[a, b]$ made of at most $m+1$ pieces of polynoms of degree at most $m+1$ whose $(m+1)^{th}$-derivative have all the same absolute value.

Since i) a perfect m-spline is m-Lipschitzian
ii) any absolutely continuous is differentiable almost everywhere
iii) any Lipschitz function f is absolutely continuous and satisfies
$L^0(f) = esssup \mid f'(x) \mid$, it follows from Glaeser's result that u is also the unique minimal m-Lipschitzian extension of T on $[a, b]$.

Moreover, for $a \leq c < d \leq b$, $u \mid_{[c,d]}$ is still a perfect spline. Applying Glaeser's result to interval $[c, d]$, it follows that $u \mid_{[c,d]}$ is also the unique minimal m-Lipschitz extension of $U \mid \{c, d\}$ on interval $[c, d]$, where U denotes the m-Whitney's function canonically associated with u.

Now we consider the general case where $dom(T)$ is any non-empty closed subset of $\mathbb{R}$. Since $\mathbb{R} - dom(T)$ is a finite or countable union of intervals $]a_i, b_i[$, $a_i < b_i$, $a_i, b_i \in dom(T)$ (or $a_i = -\infty$, $b_i \in dom(T)$, or $a_i \in dom(T)$, $b_i = +\infty$), we extend T on each interval $]a_i, b_i[$ by the Glaeser's perfect spline if a_i, b_i are finite and by $T_{a_i}(x)$ if $b_i = +\infty$, $T_{b_i}(x)$ if $a_i = -\infty$.

Using the considerations developped in the case $dom(T) = \{a, b\}$ and the glueing property of Lipschitz constants (see [[4] a tour ,p.60]) we obtain the following theorem which is just a reformulation of Glaeser's result in terms of Lipschitz minimality.

Theorem 4.1. *Let T be a Lipschitzian m-Whitney's function. Then the extension u just constructed is the unique absolutely minimal m-Lipschitzian extension of T in the following sense : for any open subset D of $\mathbb{R}$, for any v extending $U \mid \partial D$ where U denotes the m-Whitney's function associated with u , we have*
$$L^0(v^{(m)} \mid D) \geq L^0(u^{(m)} \mid D) \ .$$

Now let us turn to the multivariate case $m = 1$, $d = 1$, any $n > 1$. The concept of absolutely minimizing function introduced by G. Aronsson in [2], carried over to the case of Lipschitz functions in [1] can also be carried over to the differentiable case as follows.

Definition 4.2. Let Ω be an open non empty subset of $\mathbb{R}^n$. A 1-Lipschitzian function u defined on Ω is said to be absolutly minimal in Ω if $L^1(U \mid \bar{D}) = L^1(U \mid \partial D)$ for any bounded open subset D of Ω satisfying $\bar{D} \subset \Omega$. Here, as usual, U denotes the 1-Whitney's function associeted to u :

$$U_x(y) := u(x) + < Du(x), y - x > \text{ for any } x \in \Omega, y \in \mathbb{R}^n.$$

Drawing a parallel with the continuous case studied by G. Aronsson, it is plausible that absolutely minimal functions are correlated with functions which minimize the *sup norm* of the Hessian $H_x(u) := (u_{x_i x_j})_{1 \leq i,j \leq n}$ where $x = (x_1, ..., x_n)$, that is $\sup_{x \in \Omega} ||| H_x(u) |||$ where

$$||| H_x(u) ||| := sup_{h \neq 0 \in \mathbb{R}^n} \frac{\| H_x(u)h \|}{\| h \|} = sup\{| \lambda / \ \lambda \text{ eigenvalue of } H_x(u)\}.$$

Writing $\mu := ||| H_x(u) |||$, we obtain by purely formal calculus of variation the following PDE (non linear and of order 3) :

$$< AV, V >= 0 \ ,$$

where

$$A = (\frac{\partial \mu}{\partial u_{x_i x_j}})_{1 \leq i,j \leq n} \ , \ V = (\frac{\partial \mu}{\partial x_i})_{1 \leq i \leq n}.$$

This heuristic correlation remains to be mathematically established.

References

[1] ARONSSON, G., *Extension of functions satisfying Lipschitz conditions*, Ark. Math. 6, (1967), pp. 551-561.

[2] ARONSSON, G., *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$, Ark. Math. 6, (1965), pp. 33-53.

[3] ARONSSON, G., *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$. *II.*, Ark. Math. 6, (1966), pp. 409-431.

[4] ARONSSON, G., GRANDALL, M., AND JUUTINEN, P., *A tour of the theory of Absolutely Minimizing Functions*, http:www.math.ucsb.edu, July (2003).

[5] COATMELEC, Chr., *Approximation et interpolation des fonctions différentibles de plusieurs variables*, Ann. scient. Éc. Norm. Sup. t. 83, (1966), pp. 271-341.

[6] FEDERER, H., *Geometric Measure Theory*, Springer-Verlag New York, (1969), pp. 199-202.

[7] GLAESER, G., *Études de quelques algèbres tayloriennes*, J. d'analyse math. vol. 6, (1958) pp. 1-124.

[8] GLAESER, G., *Prolongement Extremal de Fonctions Différentiables d'une variable*, J. Approx. Math. Theory, 8, 1973, p. 249-261.

[9] HESTENES, M.R., *Extension of the range of differentiable functions*, Duke. Math. Journal (1941), pp 183-192.

[10] JENSEN, R., *Uniqueness of Lipschitz extensions: minimizing the sup-norm of the gradient*, Arch. Rational Mech. Anal. 123, (1993), no. 1 pp. 51-74.

[11] KIRSZBRAUN, M.D., *Über die zusammenziehenden und Lipschitzschen Transformationen*, Fund. Math. vol. 22 (1934), pp 77-108.

[12] McSHANE, E.J., *Extension of range of functions*, Bull. Amer. Math. Soc., 40, (1934), p. 837-842.

[13] WHITNEY, H., *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc., 36, (1934), no. 1, p. 63-89.